

THREE CLASSES OF TRIANGLES

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*Сочинились стихи бредовые
Без какой-либо путной мысли,
Словно ведра воды пудовые
Я принес вам на коромысле.
А.Великий*

1 Definitions and introductory problems

Every triangle has an incircle and a circumcircle. In addition to it, every triangle has three excircles, each of them touching one of the sides of the triangle and the extensions of two other sides. For a triangle ABC we denote:

Ω — circumcircle, O , R — its center and radius;

γ — incircle, I , r — its center and radius;

$\gamma_a, \gamma_b, \gamma_c$ — excircles, I_a, I_b, I_c — their centers, r_a, r_b, r_c — their radii.

The main objects that we study are three special kinds of triangles.

Definition A. *A-triangle* is a triangle in which $OI \parallel BC$.

Definition B. *B-triangle* is a triangle in which circles γ_b и Ω are perpendicular.

Definition C. *C-triangle* is a triangle in which $R = r_c$.

In the problems that are marked by the letters A, B, C , we mean that A, B, C -triangle is given (otherwise the triangle ABC is arbitrary). If we formulate some proposition in the problem, you should prove it. For every proved property A, B, C -triangles we offer you to find out, if it is equivalent to the definition of A, B, C -triangles.

For the triangle ABC we use such notations:

C_0, A_0, B_0 — midpoints of AB, BC и CA respectively;

AH_a, BH_b, CH_c — altitudes;

H'_a, H'_b, H'_c — second intersection points of altitudes and Ω .

C', A', B' — middles of the arcs AB, BC, CA of the circle Ω , not containing its vertices;

C'', A'', B'' — middles of the arcs ACB, BAC, CBA of the circle Ω ;

C_1, A_1, B_1 — intersection points of γ and the sides AB, BC, AC respectively;

C_c, A_c, B_c — intersection points of γ_c and the sides AB, BC, AC (similarly for the circles γ_a и γ_b);

H — orthocenter, M — the common point of medians, G — the Gergonne point (the common point of AA_1, BB_1, CC_1), N — The Nagel point (the common point of AA_a, BB_b, CC_c), F — the Feuerbach point.

1. In any acute triangle the sum of distances from O to the sides equals $R + r$.
2. a) In any triangle $OI^2 = R^2 - 2Rr$.
In any triangle $OI_c^2 = R^2 + 2Rr_c$.
3. Let KA_1 — be the diameter of the circle γ . Then AK contains N .

- 1A. Construct A -triangle if its circumcircle and the point I are given.
- 2A. Point A' is the center of excircle of triangle Δ , whose sides are the lines OI , AB and AC .
- 3A. N lies on AO .
- 4A. $\cos \angle B + \cos \angle C = 1$.
- 5A. $r + r_a = 2R$.
- 6A. A_0I intersects AH at the point Z such that $AZ = r$.
- 7A. $|OA_1| = R - r$.
- 8A. A_1 lies on H'_aO .
- 9A. $\angle AIH = 90^\circ$.
- 10A. F lies on AH .
- 11A. The center of homothety with positive ratio taking γ to Ω lies in AH_a .
- 12A. $A'C'$ и $A'B'$ — are external angle bisectors of Δ .
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- 1B. $r_b = 2R$.
- 2B. а) $\cos \frac{A}{2} \cos \frac{C}{2} \sin \frac{B}{2} = \frac{1}{2}$.
 б) $\cos C + \cos A = \cos B + 1$.
- 3B. H lies on A_1C_1 .
- 4B. The line that passes through the bases of the altitudes drawn from A и C , is tangent γ .
- 5B. The point symmetric to I with respect to O , lies on BC .
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- 1C. Determine the length of the common chord of γ_c и Ω (R is given).
- 2C. Find the angles of isosceles C -triangle
- 3C. а) $\cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} = \frac{1}{4}$.
 б) $\cos C = \cos A + \cos B$.
- 4C. Let C_c , A_c , B_c — be the tangent points of γ_c with the side AB and the extensions of the sides BC , AC .
- 5C. а) Prove that the segments IC_c , I_aA_c , I_bB_c и I_cO have a common point.
 б) Determine the ratio in which this point divides the segment I_cO .

6C. $AA_c = BB_c = CC_c$.

It is known that lines joining vertices of a triangle with touching points of the inscribed circle and the opposite side have a common point. This point is called G is called *Gergonne point*. Similarly, if we replace the incircle by the excircle we can define three *external Gergonne points* G_a, G_b, G_c .

7C. G_c lies on the circumcircle of the triangle ABC .

8C. O — is the orthocenter of triangle $A_cB_cC_c$.